The Fubini Principle

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Mathematical tools usually have names, be they well-established theorems or useful formulas. Others, as Sherman Stein writes, “serve anonymously,” though they are not less important. Here we consider one of them—the Fubini principle. No preliminary explanation seems better than a simple example, so we proceed to a problem right away.

Problem 1 (IMO, 1987). Let \( p_n(k) \) be the number of permutations of the set \( \{1, 2, \ldots, n\} \) which have exactly \( k \) fixed points. Prove that

\[
\sum_{k=0}^{n} kp_n(k) = n!
\]

for each positive integer \( n \).

Solution. Consider a rectangular table with \( n! \) rows and \( n \) columns. Label all \( n! \) permutations of \( \{1, 2, \ldots, n\} \) with the numbers \( 1, 2, \ldots, n! \). We write 1 at the intersection of row \( i \) and column \( j \) if \( j \) is a fixed point of permutation \( i \), and 0 otherwise.

Let us count the ones on the table by rows. By the definition of the numbers \( p_n(k) \), for each \( k = 0, 1, \ldots, n \) there are exactly \( p_n(k) \) rows containing \( k \) ones. Hence the sum \( S \) of all ones is equal to \( \sum_{k=0}^{n} kp_n(k) \).

We now count the ones by columns. The number of ones in the \( j \)th column is exactly the same as the number of permutations for which \( j \) is a fixed point. These permutations are easy to count: there are no restrictions on their actions on the remaining elements \( 1, \ldots, j-1, j+1, \ldots, n \). It follows that the number of permutations leaving \( j \) unchanged is \( (n-1)! \), so that the sum of the ones, counted by columns, equals

\( S = n \cdot (n-1)! = n! \).

This proves the claim.

Problem 2 (IMO, 1998). In a contest, there are \( a \) candidates and \( b \) judges, where \( b \geq 3 \) is an odd integer. Each candidate is evaluated by each judge as either passing or failing. Suppose that each pair of judges agrees on at most \( k \) candidates. Prove that

\[
\frac{k}{a} \geq \frac{b - 1}{2b}.
\]

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Solution. Here it proves appropriate to consider the total number $S$ of agreements between judges.

Since there are $\binom{b}{2}$ pairs of judges and each pair agrees on at most $k$ candidates, we have

$$S \leq k \binom{b}{2}. \quad (1)$$

Let the $i$th candidate ($i = 1, 2, \ldots, a$) be passed by $x_i$ judges and failed by $y_i$ judges. Then the number of pairs of judges who agree on this candidate is

$$\binom{x_i}{2} + \binom{y_i}{2} = \frac{1}{2}(x_i^2 + y_i^2 - x_i - y_i)$$

$$\geq \frac{1}{2} \left[ \frac{1}{2} (x_i + y_i)^2 - b \right] = \frac{1}{4} [(b - 1)^2 - 1].$$

Since $b$ is odd, this lower bound can be strengthened to $\frac{1}{4} (b - 1)^2$. It follows that

$$S = \sum_{i=1}^{a} \left[ \binom{x_i}{2} + \binom{y_i}{2} \right] \geq \frac{a(b-1)^2}{4}. \quad (2)$$

Comparing (1) and (2), we obtain

$$k \binom{b}{2} \geq \frac{a(b-1)^2}{4},$$

yielding the desired conclusion.

**Problem 3.** We call a rectangle *semi-integer* if at least one of its sides has integer length. Prove that if a rectangle $R$ can be divided into a finite number of semi-integer rectangles then $R$ is itself semi-integer.

**Solution.** The problem is exceptionally interesting and has a multitude of spectacular solutions [1]. We consider only one involving the Fubini principle.

Assume a system of rectangular coordinates $Oxy$ is given in the plane so that the rectangle $R = OABC$ is placed as shown in the figure, with its lower-left-hand corner at the origin. Assume that it is divided into finitely many semi-integer rectangles $R_1, R_2, \ldots, R_n$. Denote by $S$ the set of their vertices which have integer coordinates. For each point $V$ of $S$ let $v(V)$ be the number of rectangles among $R_1, R_2, \ldots, R_n$ of which $V$ is a vertex. We explore all possibilities:

(a) $V$ is one of the points $O, A, B, C$ and then $v(V) = 1$.

(b) $V$ is an interior point for a side of $R$ (like the point $V_1$ in the figure), in which case $v(V) = 2$.

(c) $V$ is in the interior of $R$. This is possible only if:

- $V$ is a common vertex of two rectangles and an interior point for a side of a third rectangle (like the point $V_2$ in the figure); we then have $v(V) = 2$; or
- $V$ is a common vertex of four rectangles (like the point $V_3$ in the figure); then $v(V) = 4$. 

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Now assign to each rectangle $R_i$ the number $\sigma(R_i)$ of its vertices that are in $S$, i.e., the ones having integer coordinates. Clearly, the contribution of a point $V \in S$ to the sum

$$\sum_{i=1}^{n} \sigma(R_i)$$

equals the number of rectangles among $R_1, R_2, \ldots, R_n$ that have $V$ as a vertex, that is, $\nu(V)$. Hence

$$\sum_{i=1}^{n} \sigma(R_i) = \sum_{V \in S} \nu(V). \quad (3)$$

Note now that since $R_i$ has at least one side of integer length, $\sigma(R_i)$ is even. More exactly, $\sigma(R_i)$ is equal to 0, 2 or 4. It follows that the left-hand side of (3) is even, and thus so is the right-hand side. But one of the summands on the right—the number $\nu(O)$—equals 1. Consequently, at least one more $\nu(V)$ is odd for some point $V \in S$. However, we checked that this is possible only if $V$ is one of the vertices $A$, $B$, $C$ of the big rectangle $R$. So at least one vertex of $R$ other than $O$ has integer coordinates, implying that $R$ has a side of integer length. \[\square\]

A direct consequence of this result is a nontrivial necessary and sufficient condition for cutting a rectangle into congruent smaller rectangles. We omit the immediate proof.

An $a \times b$ rectangle can be cut into $c \times d$ rectangles if and only if one of the following condition holds:

(a) one of the numbers $a$ and $b$ is an integer multiple of $c$, and the other is an integer multiple of $d$;

(b) one of the numbers $a$ and $b$ is an integer multiple of both $c$ and $d$, and the other can be written in the form $ma + nb$ for some positive integers $m$ and $n$.

**Problem 4.** A solitaire game is played on an $m \times n$ rectangular board, using $mn$ markers that are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but one must then turn over all markers that are in squares having an edge in common with the square of the removed marker. Determine all pairs $(m, n)$ of positive integers such that all markers can be removed from the board.
This is one of the hardest problems shortlisted for the jury of the international mathematical olympiad in Taiwan in 1998.

Solution. Trying small values of \( m \) and \( n \), it is not hard to guess that all markers can be taken away if and only if at least one of the numbers \( m \), \( n \) is odd. Here is a rigorous proof.

Let, for instance, \( m = 2k - 1 \). We denote the markers by the coordinates \( (i, j) \) of their locations where \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). One may assume that \( (1, 1) \) has its black side up. Remove the markers \( (i, 1) \) in the order \( i = 1, 2, \ldots, m \). We are done, if \( n = 1 \). Suppose that \( n \geq 2 \). Then each marker \( (i, 2) \) has its black side up. We remove \( (2i - 1, 2) \) in the order \( i = 1, 2, \ldots, k \). Now each \( (2i, 2) \) has changed color twice, so they can be taken away, independently of one another. This completes the task if \( n = 2 \). Otherwise, each \( (i, 3) \) has its black side up, and the procedure can be repeated.

\[
\begin{array}{c|c|c|c|c}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 \\
16 & 17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 & 25 \\
\end{array}
\]

We proceed to the converse. This is the harder part of the proof, which can be carried out in a number of ways. We present an argument using the Fubini principle.

Assume that all markers can be taken away. As you remove each one of them, write down the number of its neighbors which have already been taken away from the board, an integer among 0, 1, 2, 3, 4. Let \( S \) be the sum of all such numbers. We express \( S \) in two different ways and then compare the results, using a parity argument.

The first removed marker contributes 0 to \( S \). Each marker removed after this was initially white. Since at the moment of its removal it is black, an odd number of its neighbors must have been taken away before this. Hence the contribution of this marker to \( S \) is an odd number. We obtain that \( S \) is a sum of \( mn - 1 \) odd integers, and so

\[ S \equiv mn - 1 \pmod{2}. \]

Now consider a pair of neighboring markers. Whichever of the two is removed first contributes 1 to \( S \) when the other one is eventually taken away. Thus \( S \) is equal to the number of pairs of adjacent squares on the board, which is readily seen to be \( m(n - 1) + n(m - 1) \):

\[ S = m(n - 1) + n(m - 1). \]

Therefore a necessary condition for removing all markers is that \( mn - 1 \) and \( m(n - 1) + n(m - 1) \) be of the same parity. This is equivalent to \((m - 1)(n - 1)\) being even, that is, at least one of \( m \) and \( n \) being odd. This completes the proof. ■

REFERENCES


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